NON-STANDARD METHOD OF SOLVING DIRECT AND INVERSE PROBLEMS FOR HYPERBOLIC EQUATIONS

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Abstract. We consider a non-standard method that has been used for solving parabolic heat equations, but never to solve hyperbolic equations describing oscillatory processes. This technique was developed by Abraham Temkin (1919-2007) in the 1960s and the concept summary is described in the monograph by A. Temkin, "Inverse Methods of Heat Conduction", Moscow: Energija Press, 1973; 464 p. (in Russian). The method is based on the fact that for non-stationary heat conduction with non-stationary boundary conditions, the influence of initial conditions on the temperature distribution decreases. And after a while, one can assume that the temperature distribution is determined only by a change of boundary conditions over time. Hyperbolic equations have the same property, so it is useful to check whether this method applies to hyperbolic equations. When applying Temkin's method, we seek a solution in the form of a series where each term is a product of a derivative of the given boundary condition and an unknown function P of a space variable. Plugging the series into the given differential equation yields a system of ordinary differential equations. When solving this, we find the spatial functions P. Further, we compare the classical solution with the solution obtained by this method. The spatial functions are either polynomials or expressions that contain a polynomial as an addend, depending on the geometry of the domain and the type of the boundary conditions. Such a solution allows us to formulate the inverse problem to find the speed of propagation, knowing amplitudes of oscillations at an intermediate point of the domain. The method proposed here allows us to obtain simple formulas for approximate solution of the inverse problem.

Keywords: hyperbolic equation, non-standard method, direct problem, inverse problem.

Introduction

This paper deals with the direct problem for hyperbolic equation and the inverse problem of determination of the speed of the propagation coefficient which describes the characteristics of the environment in which the oscillations occur. There are many situations when this unknown coefficient cannot be measured directly. Thus, some mathematical techniques are required to estimate the wave speed indirectly. Many authors have studied the coefficient inverse problems for hyperbolic equations, and a number of approaches have been proposed and developed, e.g. [1-7]. These techniques have various practical applications in many areas of science like geoscience, physics and engineering. For example, Goncharskii and Romanov [3] consider two approaches for solving coefficient inverse problems. The methods developed there are intended to find inhomogeneities in homogeneous media and can be applied for solving problems in medical diagnostics, in acoustics and seismology, etc. (see [3] for more).

In this work, our attention is concentrated on the case when waves propagate in a one-dimensional medium with a periodic boundary condition imposed. The geometry of the environment is chosen as simple as possible to provide the simplest mathematical model of the process. The same kind of approach was used by A. Temkin [8] for different types of inverse problems.

The first section covers the model of a direct problem when using Temkin's method to solve the given problem. In the second section it is suggested how to use this solution in determining the speed of propagation. That is an inverse problem, where the coefficient can be determined on the basis of some experimental measurements, for example, from internal measurements of the amplitudes of oscillations. Instead of experimental data, we use the results of the classical model of wave equation solved by the Fourier method, as in [9; 10]. Lastly, some numerical results are presented.

Mathematical model of direct problem

Suppose that the wave propagation with a as the speed of propagation is described by this homogeneous one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \, x \in (0, l), \, t > 0 \tag{1}$$

where the following conditions are prescribed: for the initial conditions we have

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0, x \in (0,l) \tag{2}$$

and the boundary conditions state that

$$u(0,t) = 0, u(l,t) = u_1(t), t > 0 \tag{3}$$

We are going to reformulate the problem (1)-(3) in terms of non-dimensional variables. So, we introduce a new coordinate ξ instead of x, and τ instead of t:

$$\xi = \frac{x}{l^*}, \tau = \frac{t}{t^*}, v(\xi,\tau) = \frac{u(x,t)}{u^*} \tag{4}$$

In dimensionless variables (4), equation (1) reads as

$$\frac{\partial^2 v}{\partial \tau^2} = \frac{a^2(t^*)^2}{(l^*)^2}\frac{\partial^2 v}{\partial \xi^2}.$$

This suggests choosing $t^* = \frac{l^*}{a}$, so that $\frac{a^2(t^*)^2}{(l^*)^2} = 1$. For l^* we are going to take the length of the interval, i.e., $l^* = l$. Now the problem becomes:

$$\frac{\partial^2 v}{\partial \tau^2} = \frac{\partial^2 v}{\partial \xi^2}, \xi \in (0,1), \tau > 0 \tag{5}$$

where $v(\xi,\tau)$ satisfies the initial conditions:

$$v(\xi,0) = 0, \frac{\partial v}{\partial \tau}(\xi,0) = 0, \xi \in (0,1) \tag{6}$$

and the boundary conditions:

$$v(0,\tau) = 0, v(1,\tau) = \frac{u_1(t)}{u^*} \equiv v_1(\tau), \tau > 0 \tag{7}$$

Temkin's method begins (as described in [8] and used in [11; 12]) by assuming a separable solution of the form

$$v(\xi,\tau) = \sum_{n=0}^{\infty} P_n(\xi)T_n(\tau) \tag{8}$$

where $T_n(\tau)$ represents the n^{th} derivative of the given boundary condition at $\xi = 1$, i.e.,

$$T_n(\tau) = \left(v_1(\tau)\right)^{(n)} \text{ or } \frac{\partial^n v_1(\tau)}{\partial \tau^n}$$

but P_n are unknown functions depending on the space variable ξ. Formula (8) does not contain the initial conditions although (1) admits a unique solution if and only if the initial conditions are in the form (2). When applying this method, the initial conditions are used and those are

$$v(\xi,0) = \sum_{n=0}^{\infty} P_n(\xi)T_n(0)$$

$$\frac{\partial v}{\partial \tau}(\xi,0) = \sum_{n=0}^{\infty} P_n(\xi)\frac{\partial T_n}{\partial \tau}(0)$$

So,

$$\frac{\partial v}{\partial \tau}(\xi,0) = \sum_{n=0}^{\infty} P_n(\xi)T_{n+1}(0)$$

Plugging the form (8) into the wave equation (5), the partial differential equation is transformed into ordinary differential equations for $P_n(\xi)$, and after imposing the boundary conditions (7), they give

$$P_0(\xi) = \xi$$

$$P_1(\xi) = 0$$

$$P_2(\xi) = \frac{\xi^3}{6} - \frac{\xi}{6}$$

$$P_3(\xi) = 0$$

$$P_4(\xi) = \frac{\xi^5}{120} - \frac{\xi^3}{36} + \frac{7\xi}{360}$$

$$P_5(\xi) = 0$$

$$P_6(\xi) = \frac{\xi^7}{5040} - \frac{\xi^5}{720} + \frac{7\xi^3}{2160} - \frac{31\xi}{15120}$$

$$P_7(\xi) = 0 \text{ etc.} \tag{9}$$

The dimensionless solution to (5)-(7) is

$$v(\xi,\tau) = \frac{u(\xi \cdot l, t)}{u^*} = \sum_{n=0}^{\infty} P_n(\xi) \cdot \frac{1}{u^*} \cdot \frac{\partial^n u_1(t)}{\partial t^n} \cdot (t^*)^n$$

Therefore, we have a solution of (1) -(3):

$$u(x,t) = u(\xi \cdot l, t) = \sum_{n=0}^{\infty} P_n(\xi) \cdot u_1^{(n)}(t) \cdot \left(\frac{l}{a}\right)^n \tag{10}$$

As the convergence of the series is not considered in this paper, further investigation must be done on the subject.

We shall assume that the boundary condition at $x = l$ is periodic and defined as

$$u(l,t) = A\sin(\omega t) \tag{11}$$

Using formulas given in [9; 10], the problem (1)-(3) with boundary condition (11) has a solution:

$$u(x,t) = A\frac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\sin\omega t +$$

$$+ \frac{2A\omega a}{l}\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{\omega^2 - \left(\frac{k\pi a}{l}\right)^2}\sin\frac{k\pi a t}{l}\sin\frac{k\pi x}{l},\ \omega \neq \frac{k\pi a}{l} \tag{12}$$

The difference between these two solutions obtained by both methods is given in Fig. 3.

Mathematical model of inverse problem

The inverse problem is formulated as a coefficient inverse problem – we want to determine the speed of propagation a, knowing amplitudes of oscillations (for example, from experimental data) at an intermediate point of the domain. Let us call this point ξ^*, where $\xi^* \in (0,1)$.

To obtain the solution using Temkin's method, we used an infinite sum in expressing it, i.e., (10). When taking a finite number of terms from the series, the approximate solution is obtained. This can be written as

$$u(\xi^* \cdot l, t) = \sum_{n=0}^{M} P_n(\xi^*) \cdot u_1^{(n)}(t) \cdot d^n, \text{ where } d = \frac{l}{a} \tag{13}$$

Solving this equation for a^2 with different numbers of terms in the series might result in useful approximation of the speed of propagation a. The approximation of a^2 might become better and better as more and more terms are included.

Starting with $M = 2$ in (13) produces

$$P_0(\xi^*)u_1(t) + P_1(\xi^*)u_1'(t)d + P_2(\xi^*)u_1''(t)d^2 = u(\xi^* l, t)$$

As $P_1(\xi^*) = 0$ (see (9) for more), this yields a linear equation for d^2. After solving the obtained linear equation, we receive the first approximation to a^2:

$$a^2 = \frac{l^2 P_2(\xi^*)u_1''(t)}{u(\xi^* l,t) - P_0(\xi^*)u_1(t)} \tag{14}$$

Given an approximate expression of a^2, a closer approximation can be found by increasing the terms in the sum (13). If $M = 4$, the truncated sum reduces to

$$P_4(\xi^*) \cdot u_1^{(4)}(t) \cdot d^4 + P_2(\xi^*) \cdot u_1''(t) \cdot d^2 + P_0(\xi^*) \cdot u_1(t) = u(\xi^* l, t)$$

With notation $b = d^2$, this equation becomes quadratic

$$P_4(\xi^*) \cdot u_1^{(4)}(t) \cdot b^2 \; + \; P_2(\xi^*) \cdot u_1''(t) \cdot b \; + \; P_0(\xi^*) \cdot u_1(t) \; = \; u(\xi^* l, t) \tag{15}$$

the solution to which is

$$b \; = \; \frac{-\left(P_2(\xi^*)\cdot u_1''(t)\right) \pm \sqrt{\left(P_2(\xi^*)\cdot u_1''(t)\right)^2 - 4P_4(\xi^*)\cdot u_1^{(4)}(t)\left(P_0(\xi^*)\cdot u_1(t) - u(\xi^* l, t)\right)}}{2P_4(\xi^*)\cdot u_1^{(4)}(t)} \tag{16}$$

with $b \; = \; \frac{l^2}{a^2}$. As formula (15) is quadratic polynomial of second order, it may have 0 or 2 real roots. You choose the one that is closer to (14). To obtain a better approximation, more terms should be included in the truncated sum.

Results and discussion

We have obtained some numerical results for the direct problem (1) - (3) with boundary condition (11) using solutions (10) and (12), and for the inverse problem as well. In the numerical experiment, we chose these parametric values: $a \; = \; 0.01 \; \frac{m}{s}$, $l \; = \; 0.02\, m$, $A \; = \; 2$, $\omega \; = \; 0.01$ and considered the problem on the time interval $t \in [0.650]$.

First, those in Fig. 1 are the functions $P_n(\xi)$ dependent on the dimensionless spatial variable $\xi \in [0,1]$. As you can see in the graphs below, these functions approach zero, as n becomes larger.

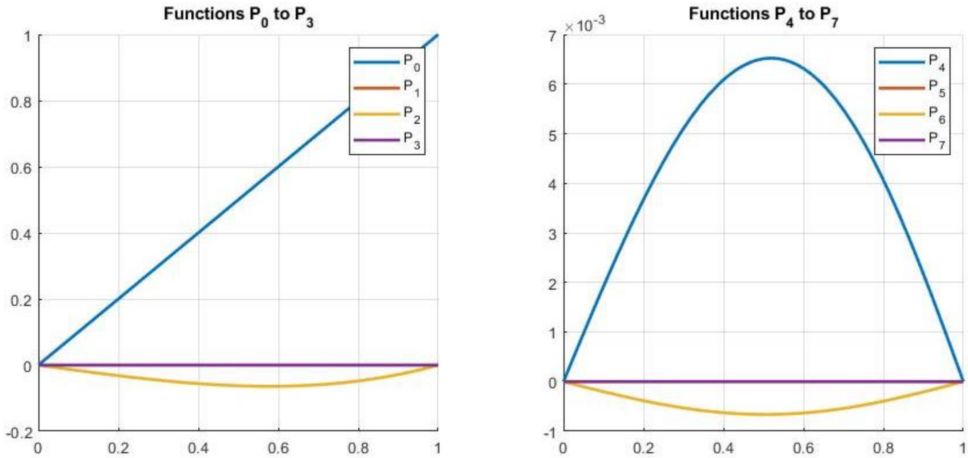

Fig. 1. **Functions P_n**

Now consider the function $u(x,t)$ graphed in Fig. 2. This is the solution of the initial boundary value problem when taking the first seven terms in the sum (10).

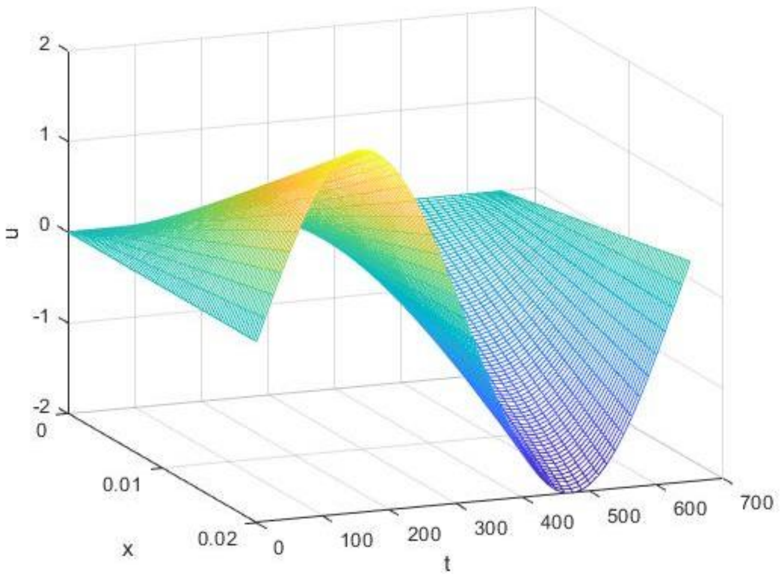

Fig. 2. **Solution of direct problem using Temkin's method**

The graph of the function (10) has the same shape as the graph of the function (12), which is the classical solution. When comparing the two we used the difference between these two solutions:

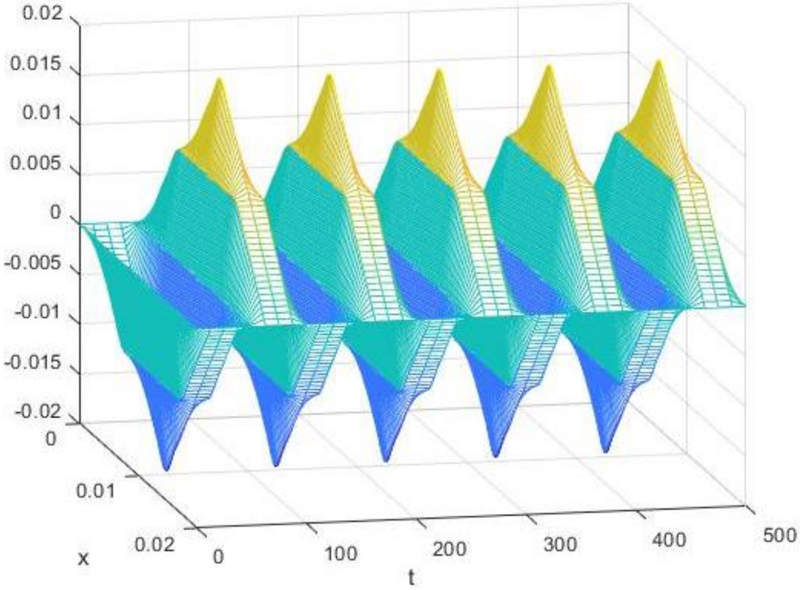

Fig. 3. **Difference between solutions (10) and (12)**

Lastly, Table 1 presents some information on the inverse problem when determining the coefficient of speed propagation for $t = 500$ and calculating the percent relative error.

Table 1

Approximation for a^2, percent relative errors

Spatial coordinate	Relative error using (14), %	Relative error using (16), %
$\xi^* = \frac{1}{4}$ or $x^* = \frac{l}{4}$	0.004541650089739	$1.896810481035946 \cdot 10^{-7}$
$\xi^* = \frac{1}{2}$ or $x^* = \frac{l}{2}$	0.004166662416990	$1.693541953808747 \cdot 10^{-7}$
$\xi^* = \frac{3}{4}$ or $x^* = \frac{3l}{4}$	0.003541679657014	$1.384199151783713 \cdot 10^{-7}$

Conclusions

Even the method developed by A. Temkin has never before been used for hyperbolic equations, the results show that it can be applied to wave equations as well. You can use the series for the inverse problem as well. However, there is a need for further investigation to cover some additional topics that were not addressed in this paper. For example:

1. The convergence of the series (10) must be shown.
2. The root-finding procedure for determining the approximation of a^2 should describe which of the real roots of the polynomial is the right one and what appropriate steps should be taken in the case of complex roots.
3. The considered method is not well known and there is no information on this method being used by scientists living abroad. An advantage of this method is that it is much simpler compared to other methods that are used for solving inverse hyperbolic equations.

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Author contributions

Conceptualization, Iltins I.; software, Treilande T.; formal analysis and investigation, Treilande T., Iltins I., writing-original draft preparation, Treilande T., writing-review and editing, Iltins I. All authors have read and agreed to the published version of the manuscript.

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